References

[1] J. E. Cruz and R. L. Brooke, "A variable characteristic impedance coaxial line," *IEEE Trans. on Microwave Theory and Techniques* (*Correspondence*), vol. MTT-13, pp. 477–478, July 1965.

[2] T. S. Chen, "Determination of the capacitance, inductance, and characteristic impedance of rectangular lines," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-8, pp. 510–519, September 1960.

[3] J. J. Skiles and T. J. Higgins, "Determination of the characteristic impedance of UHF coaxial rectangular transmission lines," *Proc. Nat'l Electronics Conf.* (Chicago, Ill.), vol. 10, pp. 97–108, October 1954.

[4] O. R. Cruzan and R. V. Garver, "Characteristic impedance of rectangular coaxial transmission lines," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-12, pp. 488–495, September 1964.

[5] S. Ramo, J. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics*. New York: Wiley, 1965 (general analysis of guided waves, pp. 393–400).

[6] R. L. Brooke, C. A. Hoer, and C. H. Love, "Inductance and characteristic impedance of strip-lines," *J. Research NBS* (*Engineering and Instrumentation*), vol. 71C, pp. 59–67, January–March 1967.

[7] R. Carlson, S. Krakauer, R. Monnier, V. Van Duzer, and R. Woodbury, "Sampling oscillography," Hewlett-Packard Applications Note 36, November 1959.

[8] "Transmission line testing using the sample oscilloscope," Hewlett-Packard Applications Note 53.

[9] "Time-domain reflectometry," Hewlett-Packard Applications Note 62, 1964.

[10] "Pulse reflection measurement of transmission line impedance and discontinuities," in-house publication, Tektronix, 1962.

Radiation from an Infinite Array of Parallel-Plate Waveguides with Thick Walls

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Abstract—**A semi-infinite array of parallel-plate waveguides with walls of finite thickness is excited by incident TEM modes in every waveguide identically. By proper application of the boundary conditions, two Wiener-Hopf equations are obtained which, however, cannot be solved by the standard techniques. A method originated by Jones [6] is applied to recast these two equations so that the forms of the solutions are found. The solutions involve constants to be determined by an infinite set of linear simultaneous equations which converge absolutely. When the thickness of the walls b is small compared with the wavelength λ, explicit solutions in the order of $0(b/\lambda)$ are found in very simple forms.**

I. Introduction

THE PROBLEM of radiation from an infinite array of parallel-plate waveguides is of great interest theoretically and practically. In the theoretical aspect, it offers an excellent example for the study of periodic structures. In particular, it was one of the first problems solved exactly by the Wiener-Hopf technique [1]. From the practical point of view, it simulates a phased array of waveguides which is widely used in today's communication and radar systems. Wu and Galindo [2], for example, made an interesting investigation of the mutual coupling effects of phased arrays by using the solution of this problem.

Most of the analyses in connection with this problem are based on the assumption that the walls of the guides are vanishingly thin. In practice, however, walls of appreciable thickness are unavoidable. Therefore, it is desirable to study the effect of this thickness on the radiation properties. Among past works on the thick-wall probem, Epstein [3] gave an empirical correction to the case of infinitely thin wall based on experimental evidence. After an unsuccessful attempt to find a rigorous theoretical solution, Primich [4] attacked the problem by variational techniques, and obtained some results checked well by experiments. Most recently, Galindo and Wu [5] formulated the problem as an integral equation which is valid for *all* scanning angles. However, that integral equation, as stated by the authors, is nonintegrable, and numerical methods using a high-speed computer were resorted to for an approximated solution.

It is the purpose of this paper to present a solution based on the Wiener-Hopf technique for the broadside radiation of an infinite array of parallel-plate waveguides with thick walls. Particularly when the thickness of the wall is small in terms of wavelength, very simple expressions for the reflection coefficient and the radiated far field are obtained. Because of the complications and the lengthiness of the manipulations, some detailed derivations are omitted in this paper; interested readers are referred to a technical report under the same title issued by Hughes Aircraft Company [10].

II. Statement of the Problem

Consider an infinite array of parallel-plate waveguides as shown in Fig. 1. The thickness of the guide wall is b, and the width of the guide is a. The dominant TEM modes are excited inside every waveguide with equal amplitude and phase. The problem is then to find the radiated field in the

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empty half-space $z>0$, and the reflected field in the waveguides $z<0$.

From Maxwell equations, it is seen that the nonvanishing field components are H_y, E_x, and E_z which satisfy the following equations:

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k^2\right]H_y(x, z) = 0 \tag{1}$$

$$E_x(x, z) = \frac{1}{i\omega\epsilon_0}\frac{\partial}{\partial z}H_y(x, z) \tag{2}$$

$$E_z(x, z) = \frac{-1}{i\omega\epsilon_0}\frac{\partial}{\partial x}H_y(x, z) \tag{3}$$

where $k=\omega\sqrt{\mu_0\epsilon_0}$ is the free-space wave number. The time dependence $\exp -i\omega t$ is suppressed throughout this paper.

Since the incident waves in every waveguide are identical, the periodic nature of this problem allows one to consider only a unit cell, say, the cell defined by $0\leq x\leq c$. Let the incident field from the left in this cell be

$$H_y^{(i)}(x, z) = \exp ikz. \tag{4}$$

Furthermore, divide this unit cell into two regions, namely, $0\leq x\leq a$ and $a\leq x\leq c$, and let the total fields be

$$H_y^{(t)}(x, z) = \begin{cases} H_y(x, z) + H_y^{(i)}(x, z), & 0 \leq x < a \\ H_y(x, z), & a < x \leq c \end{cases} \tag{5}$$

where the scattered field $H_y(x, z)$ satisfies the wave equation given by (1). Introducing the following Fourier transform pair,

$$\phi(x, \alpha) = \int_{-\infty}^{\infty} H_y(x, z) \exp i\alpha z dz \tag{6}$$

$$H_y(x, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(x, \alpha) \exp(-i\alpha z) dz \tag{7}$$

(1) becomes, after taking the Fourier transform,

$$\left[\frac{\partial^2}{\partial x^2} - \gamma^2\right]\phi(x, \alpha) = 0 \tag{8}$$

where $\gamma=\sqrt{\alpha^2-k^2}$. The complex $\alpha=\sigma+i\tau$ plane is cut as shown in Fig. 2, and the proper branch of γ is chosen such that $\gamma\rightarrow+\sigma$ as α approaches infinity along the positive real axis.

Now the problem is to solve (8) subject to the following boundary conditions:

1) The scattered tangential electric fields are zero at the guide wall, namely,

$$E_z(x, z) = 0 \quad \text{for } x = a, c; \quad \text{and} \quad z < 0$$
$$E_x(x, z) = 0 \quad \text{for } a < x < c; \quad \text{and} \quad z = 0.$$

2) The total fields are continuous at $x=a$.
3) The total fields at the plane $x=0$ are identical to the fields at the plane $x=c$.

The last condition arises from the periodic property of the structure and the excitations.

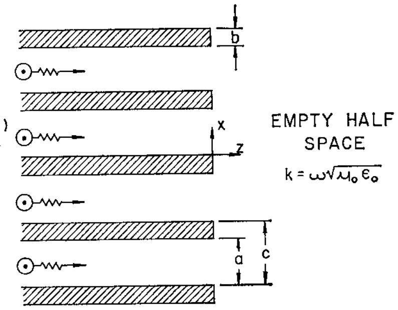

Fig. 1. Geometry of the problem.

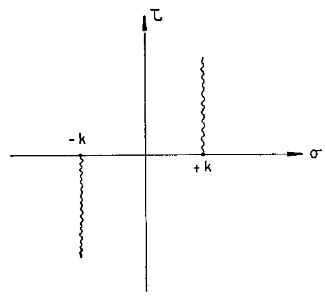

Fig. 2. Complex α-plane and branch cut.

III. Formulations of Wiener-Hopf Equations

For analytical convenience in the following discussion, let us first introduce a small loss in the medium, i.e., $k=k_1+ik_2$ where k_2 is a small positive real number. Then it is easy to show that, except for the incident field which increases exponentially as $z\rightarrow-\infty$, the reflected field inside the waveguide is attenuated at least as rapidly as $\exp(-k_2|z|)$ as $z\rightarrow-\infty$. Outside the waveguide only the radiated field exists, and its is attenuated as $\exp(-k_2 z)$ as $z\rightarrow\infty$. It follows that the transformed wave equation (8) is valid only within a strip in the complex α-plane defined by $|\tau|<k_2$. Next introduce the standard Wiener-Hopf notations:

$$\phi_+(x, \alpha) = \int_0^{\infty} H_y(x, z) \exp i\alpha z dz \tag{9}$$

$$\phi_-(x, \alpha) = \int_{-\infty}^{0} H_y(x, z) \exp i\alpha z dz \tag{10}$$

where the subscript "+" signifies that $\phi_+(x, \alpha)$ is analytic in the upper α-plane defined by $\tau>(-k_2)$, while the subscript "−" signifies that $\phi_-(x, \alpha)$ is analytic in the lower α-plane defined by $\tau<k_2$. These and similar notations will be used throughout this paper. Then the solution of (8) can be written formally as, for the region $0<x<a$,

$$\phi_+(x, \alpha) + \phi_-(x, \alpha) = Ae^{-\gamma x} + Be^{+\gamma x} \tag{11}$$

where A and B to be determined are functions of α. As for the region $a<x<c$, a different transformed wave equation instead of (8) should be used. Note that

$$\int_0^{\infty}\frac{\partial^2 H_y}{\partial z^2}\exp i\alpha z dz = \left.\frac{\partial H_y}{\partial z}\right|_{z=0} - i\alpha H_y(x, 0) - \alpha^2\phi_+(x, \alpha) \tag{12}$$

where the first term in the right-hand side is proportional to E_x which is zero at $a<x<c$ and $z=0$, while the second term is unknown. Then after such a transform (1) becomes

$$\left[\frac{\partial^2}{\partial x^2} - \gamma^2\right]\phi_+(x, \alpha) = i\alpha H_y(x, 0) \quad \text{for } a < x < c. \tag{13}$$

To eliminate the unknown $H_y(x, 0)$, change the sign of α and add the resulting equation to (13). This gives

$$\left[\frac{\partial^2}{\partial x^2} - \gamma^2\right][\phi_+(x, \alpha) + \phi_+(x, -\alpha)] = 0 \quad \text{for } a < x < c \tag{14}$$

which is the desired wave equation. Its solution can be written formally as, for the region $a<x<c$,

$$\phi_+(x, \alpha) + \phi_+(x, -\alpha) = Ce^{-\gamma x} + De^{\gamma x}. \tag{15}$$

The next step is to eliminate A, B, C, and D by judiciously applying the boundary conditions. Without going into detail, the resultant two Wiener-Hopf equations are given here:

$$\frac{c}{2} H(\alpha)U_+'(\alpha) = W_-(\alpha) - \frac{\tanh(\gamma b/2)}{\gamma} U_+'(-\alpha) \tag{16}$$

$$\frac{2c}{ab\gamma^2} K(\alpha)V_+'(\alpha) = \frac{2i}{\alpha + k} + S_-(\alpha) - \frac{\coth(\gamma b/2)}{\gamma} V_+'(-\alpha) \tag{17}$$

where

$$W_-(\alpha) = V_+(-\alpha) - V_-(\alpha)$$
$$S_-(\alpha) = U_+(-\alpha) - U_-(\alpha)$$
$$V_+(\alpha) = \phi_+(c-, \alpha) - \phi_+(a+, \alpha)$$
$$V_-(\alpha) = \phi_-(0+, \alpha) - \phi_-(a-, \alpha)$$
$$U_+(\alpha) = \phi_+(c-, \alpha) + \phi_+(a+, \alpha)$$
$$U_-(\alpha) = \phi_-(0+, \alpha) + \phi_-(a-, \alpha)$$
$$U_+'(\alpha) = \frac{\partial}{\partial x}\phi_+(x, \alpha)\bigg|_{x=c-} + \frac{\partial}{\partial x}\phi_+(x, \alpha)\bigg|_{x=a+}$$
$$V_+'(\alpha) = \frac{\partial}{\partial x}\phi_+(x, \alpha)\bigg|_{x=c-} - \frac{\partial}{\partial x}\phi_+(x, \alpha)\bigg|_{x=a+}$$
$$H(\alpha) = \frac{\sinh(\gamma c/2)/(\gamma c/2)}{\cosh(\gamma a/2)\cosh(\gamma b/2)}$$
$$K(\alpha) = \frac{ab\,\gamma\sinh(\gamma c/2)}{2c\sinh(\gamma a/2)\sinh(\gamma b/2)}$$
$$\psi(a+, \alpha) = \lim_{\epsilon\to+0}\psi(x = a + \epsilon, \alpha).$$

In each of the above two equations, there are two unknowns, namely, $U_+'(\alpha)$ and $W_-(\alpha)$ in (16), and $V_+'(\alpha)$ and $S_-(\alpha)$ in (17). These two equations, however, are not of the conventional Wiener-Hopf type because of the presence of $U_+'(-\alpha)$ and $V_+'(-\alpha)$; consequently, they cannot be solved by the standard techniques. In the following section, an ingenious method originated by Jones [6] is applied so that each of the above two equations is reduced to an infinite set of simultaneous linear equations, the solution of which is discussed in Section VI.

IV. Solutions of Wiener-Hopf Equations

Let us consider (16) first. Without difficulty $H(\alpha)$ can be divided into two functions: one is analytic in the upper half α-plane defined by $\tau>(-k_2)$; the other is analytic in the lower half α-plane defined by $\tau<k_2$, namely,

$$H(\alpha) = H_+(\alpha)H_-(\alpha) \tag{18}$$

where

$$H_+(\alpha) = H_-(-\alpha) = e^{h(\alpha)} \cdot \prod_{n=1}^{\infty} \frac{\frac{ic\pi}{2n}[\alpha + i\gamma_{2nc}]}{\frac{a}{(2n-1)}[\alpha + i\gamma_{(2n-1)a}]\frac{b}{(2n-1)}[\alpha + i\gamma_{(2n-1)b}]}$$

$$h(\alpha) = -i\frac{\alpha}{2\pi}[c\ln c - a\ln a - b\ln b]$$

$$\gamma_{na} = \left[\left(\frac{n\pi}{a}\right)^2 - k^2\right]^{1/2}$$

$$\gamma_{nb} = \left[\left(\frac{n\pi}{b}\right)^2 - k^2\right]^{1/2}$$

$$\gamma_{nc}, = \left[\left(\frac{n\pi}{c}\right)^2 - k^2\right]^{1/2}.$$

It also can be shown that $H_+(\alpha)$ and $H_-(\alpha)$ behave as $|\alpha|^{-\frac{1}{2}}$ as $|\alpha|\to\infty$ in the proper half α-plane. Substituting (18) into (16), one has after certain arrangements

$$\frac{c}{2}H_+(\alpha)U_+'(\alpha) = \frac{W_-(\alpha)}{H_-(\alpha)} - \frac{\tanh\frac{\gamma b}{2}}{\gamma}\cdot\frac{U_+'(-\alpha)}{H_-(\alpha)}. \tag{19}$$

The left-hand side of (19) are all functions analytic in the upper half α-plane, and the right-hand side are all functions analytic in the lower half α-plane except for the last term. It is important to notice that $\gamma^{-1}\tanh(\gamma b/2)$ has no singularities other than simple poles in the lower half α-plane.

Thus it can be broken down in the following manner:

$$\frac{\tanh(\gamma b/2)}{\gamma} = \sum_{n=1}^{\infty} \frac{R_n}{\alpha + i\gamma_{(2n-1)b}} + M_-(\alpha) \quad (20)$$

in which R_n is the residue at the simple pole $\alpha = -i\gamma_{(2n-1)b}$ for $n = 1, 2, 3, \cdots$, and $M_-(\alpha)$ is an analytic function in the lower half α-plane, and will not be pursued here any further since its explicit form is of no direct interest to the present problem. R_n can be found easily and is given by

$$R_n = \frac{2i}{b\gamma_{(2n-1)b}}, \qquad n = 1, 2, 3, \cdots. \quad (21)$$

Substituting (20) into (19) and making proper arrangements, one obtains

$$\frac{c}{2} H_+(\alpha) U_+'(\alpha) + \sum_{n=1}^{\infty} \frac{1}{[\alpha + i\gamma_{(2n-1)b}]} \frac{2i}{b\gamma_{(2n-1)b}} \cdot \frac{U_+'(i\gamma_{(2n-1)b})}{H_-(-i\gamma_{(2n-1)b})} = + \frac{W_-(\alpha)}{H_-(\alpha)} - \frac{U_+'(-\alpha) M_-(\alpha)}{H_-(\alpha)} + \sum_{n=1}^{\infty} \frac{2i}{b\gamma_{(2n-1)b}[\alpha + i\gamma_{(2n-1)b}]} \left[\frac{U_+'(i\gamma_{(2n-1)b})}{H_-'(-i\gamma_{(2n-1)b})} - \frac{U_+'(-\alpha)}{H_-(\alpha)} \right]. \quad (22)$$

Now the left-hand side of (22) is analytic in the upper half α-plane defined by $\tau > (-k_2)$, while the right-hand side is analytic in the lower half α-plane defined by $\tau < k_2$. Since these two half-planes overlap, both sides must be equal to a polynomial $P(\alpha)$ by analytic continuation, provided that $P(\alpha)$ has the proper algebraic behavior as α tends to infinity. To determine $P(\alpha)$, the asymptotic behavior of the functions in (22) should be examined. From edge condition that $E_z \sim z^{-1}$ and $H_y \sim$ constant as $z \to +0$, and $x = 0$ or a, it follows that $U_+'(\alpha) \sim |\alpha|^{-\frac{3}{2}}$ and $W_-(\alpha) \sim |\alpha|^{-1}$ as $|\alpha|$ tends to infinity in the appropriate half-plane. Using Liouville's theorem [7], it is easy to show that $P(\alpha)$ is identically zero. Then setting the left-hand side of (22) equal to zero, one has

$$\frac{c}{2} H_+(\alpha) U_+'(\alpha) + \sum_{n=1}^{\infty} \frac{2i}{b\gamma_{(2n-1)b}[\alpha + i\gamma_{(2n-1)b}]} \cdot \frac{U_+'(i\gamma_{(2n-1)b})}{H_-(-i\gamma_{(2n-1)b})} = 0. \quad (23)$$

The above equation holds for all α. Setting $\alpha = i\gamma_{(2m-1)b}$, for $m = 1, 2, 3, \cdots$, an infinite set of simultaneous linear algebraic equations is developed, namely,

$$\frac{bc}{4} \gamma_{(2m-1)b} H_+^2(i\gamma_{(2m-1)b}) \mu_{(2m-1)} + \sum_{n=1}^{\infty} \frac{\mu_{(2n-1)}}{\gamma_{(2m-1)b} + \gamma_{(2n-1)b}} = 0 \quad (24)$$

$$\text{for } m = 1, 2, 3, \cdots,$$

where

$$\mu_{(2n-1)} = \frac{2U_+'(i\gamma_{(2n-1)b})}{b\gamma_{(2n-1)b} H_-(-i\gamma_{(2n-1)b})}, \quad \text{for } n = 1, 2, 3, \cdots.$$

This set of equations will be examined in Section VI. Now assume that $\{\mu_{(2n-1)}\}$ is found. The solution of $U_+'(\alpha)$ immediately follows from (23), i.e.,

$$U_+'(\alpha) = \frac{2}{icH_+(\alpha)} \sum_{n=1}^{\infty} \frac{\mu_{2n-1}}{\alpha + i\gamma_{(2n-1)b}}. \quad (25)$$

Next consider (17), the solutions of which can be obtained in a similar manner, as given below:

$$V_+'(\alpha) = \frac{ab}{ic} \frac{2k}{K_+(k) K_+(\alpha)} + \frac{ab(\alpha + k)}{2cK_+(\alpha)} \sum_{n=0}^{\infty} \frac{\nu_{2n}}{\alpha + i\gamma_{2nb}} \quad (26)$$

where

$$\nu_0 = \frac{-2V_+'(k)}{bK_+(k)}, \qquad \nu_{2n} = \frac{2(ik - \gamma_{2nb})}{b\gamma_{2nb} K_+(i\gamma_{2nb})} V_+'(i\gamma_{2nb}),$$

$$\text{for } n = 1, 2, 3, \cdots.$$

$$K_+(\alpha) = K_-(-\alpha) = e^{h(\alpha)} \prod_{n=1}^{\infty} \frac{2in\pi c(\alpha + i\gamma_{2nc})}{ab(\alpha + i\gamma_{2na})(\alpha + i\gamma_{2nb})}.$$

The set of undetermined constants $\{\nu_{2n}\}$ satisfies the following simultaneous linear equations:

$$\begin{cases} \dfrac{icK_+^2(k)}{2ka} \nu_0 + \displaystyle\sum_{n=0}^{\infty} \frac{\nu_{2n}}{-ik + \gamma_{2nb}} = \frac{-2}{K_+(k)} \\ \dfrac{-c\gamma_{2mb} K_+^2(i\gamma_{2mb})}{a(i\gamma_{2mb} + k)^2} \nu_{2m} + \displaystyle\sum_{n=0}^{\infty} \frac{\nu_{2n}}{\gamma_{2mb} + \gamma_{2nb}} \\ \qquad = \dfrac{4ik}{(\gamma_{2mb} - ik) K_+(k)}. \qquad m = 1, 2, 3, \cdots. \end{cases} \quad (27)$$

So far, the two unknowns $U_+'(\alpha)$ and $V_+'(\alpha)$ in the Wiener-Hopf equations are solved within two sets of constants $\{\mu_{2n-1}\}$ and $\{\nu_{2n}\}$. To determine these constants, two infinite, simultaneous linear equations must be solved. In the following section, the complete field solutions are worked out first, while the examination of the two sets of linear equations is relegated to Section VI.

V. Field Solutions

With solutions of $U_+'(\alpha)$ and $V_+'(\alpha)$ given by (25) and (27), respectively, the complete reflected field inside the waveguide and the radiated field outside the waveguide can be found without difficulty.

First consider the fields in the region $0 \leq x \leq a$. It can be shown that A and B in (11) are related to $U_+'(\alpha)$ and $V_+'(\alpha)$ in the following manner:

$$\begin{Bmatrix} A \\ B \end{Bmatrix} = \frac{1}{2\gamma} \begin{Bmatrix} \dfrac{-1}{1+e^{-\gamma a}} & \dfrac{-1}{1-e^{-\gamma a}} \\ \dfrac{1}{1+e^{\gamma a}} & \dfrac{1}{1-e^{\gamma a}} \end{Bmatrix} \begin{Bmatrix} U_+'(\alpha) \\ V_+'(\alpha) \end{Bmatrix}. \tag{28}$$

Substituting (28) into (11) and taking the inverse Fourier transform, one has

$$\begin{aligned} H_y(x, z) = {} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{U_+'(\alpha)}{2\gamma \cosh (\gamma a/2)} \cdot \sinh \gamma \left(x - \frac{a}{2} \right) e^{-i\alpha z} d\alpha \\ & - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{V_+'(\alpha)}{2\gamma \sinh (\gamma a/2)} \cdot \cosh \gamma \left(x - \frac{a}{2} \right) e^{-i\alpha z} d\alpha \\ & \text{for } 0 < x < a. \end{aligned} \tag{29}$$

As expected, there are no other types of singularities than simple poles in the integrand of (29). Evaluating the residue contributions in a straightforward manner gives the following final results:

1) For $0<x<a$ and $z \leq 0$,

$$\begin{aligned} H_y(x, z) = {} & -\frac{b}{c} \left[\frac{1}{K_+^2(k)} + \frac{i}{2} \frac{1}{K_+(k)} \sum_{n=0}^{\infty} \frac{\nu_{2n}}{k + i\gamma_{2nb}} \right] e^{-ikz} \\ & + \sum_{n=1}^{\infty} \left\{ \left[\frac{2(-1)^{n+1}}{ac\gamma_{(2n-1)a} H_+(i\gamma_{(2n-1)a})} \sum_{m=1}^{\infty} \frac{\mu_{2m-1}}{\gamma_{(2n-1)a} + \gamma_{(2m-1)b}} \right] \cos \left[\frac{(2n-1)\pi_x}{a} \right] e^{\gamma_{(2n-1)a} z} \right\} \\ & + \sum_{n=1}^{\infty} \left\{ \left[\frac{2k}{K_+(k)} + \frac{\gamma_{2na} - ik}{2} \sum_{m=0}^{\infty} \frac{\nu_{2m}}{\gamma_{2na} + \gamma_{2ma}} \right] \frac{b(-1)^{n+1} \cos \left[\dfrac{2n\pi}{a} x \right]}{c\gamma_{2na} K_+(i\gamma_{2na})} e^{\gamma_{2na} z} \right\}. \end{aligned} \tag{30}$$

2) For $0 \leq x \leq a$ and $z \geq 0$,

$$\begin{aligned} H_y(x, z) = {} & \frac{-b}{c} \left[1 + \frac{i}{4k} \frac{\nu_0}{K_-(k)} \right] e^{ikz} \\ & + \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{2n\pi i H_+'(-i\gamma_{2nc})} \sum_{m=1}^{\infty} \frac{\mu_{2m-1}}{\gamma_{2nc} - \gamma_{(2m-1)b}} \right] \frac{\sin \left[\dfrac{2n\pi}{c} \left(x - \dfrac{a}{2} \right) \right]}{\cos \left[\dfrac{n\pi}{c} a \right]} e^{-\gamma_{2nc} z} \right\} \\ & + \sum_{n=1}^{\infty} \left\{ \left[\frac{2k}{K_+(k)} + \frac{k - i\gamma_{2nc}}{2} \sum_{m=1}^{\infty} \frac{\upsilon_{2m}}{\gamma_{2mb} - \gamma_{2nc}} \right] \frac{ab \cos \left[\dfrac{2n\pi}{c} \left(x - \dfrac{a}{2} \right) \right]}{4n\pi i K_+'(-i\gamma_{2nc}) \sin \left[\dfrac{n\pi}{c} a \right]} e^{-\gamma_{2nc} z} \right\} \end{aligned} \tag{31}$$

where

$$H_+'(-i\gamma_{2nc}) = \frac{d}{d\alpha} H_+(\alpha) \bigg|_{\alpha = -i\gamma_{2nc}},$$

$$K_+'(-i\gamma_{2nc}) = \frac{d}{d\alpha} K_+(\alpha) \bigg|_{\alpha = -i\gamma_{2nc}}.$$

Next consider the radiated field in the region $a \leq x \leq c$. It can be shown that C and D in (15) are related to $U_+'(\alpha)$ and $V_+'(\alpha)$ in the following manner:

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2\gamma(e^{\gamma b} - e^{-\gamma b})} \begin{pmatrix} e^{\gamma a}(1 - e^{\gamma b}), & e^{\gamma a}(1 + e^{\gamma b}) \\ e^{-\gamma a}(1 - e^{-\gamma b}), & e^{\gamma a}(1 + e^{-\gamma b}) \end{pmatrix} \cdot \begin{pmatrix} U_+'(\alpha) + U_+'(-\alpha) \\ V_+'(\alpha) + V_+'(-\alpha) \end{pmatrix}. \tag{32}$$

Substituting (32) into (15) and taking the inverse cosine transform, one has

$$\begin{aligned} H_y(x, z) = {} & \frac{1}{2\pi} \int_0^{\infty} \frac{U_+'(\alpha) + U_+'(-\alpha)}{\gamma \cosh \dfrac{\gamma b}{2}} \cdot \sinh \left[\gamma \left(x - a - \frac{b}{2} \right) \right] \cos \alpha z d\alpha \\ & + \frac{1}{2\pi} \int_0^{\infty} \frac{V_+'(\alpha) + V_+'(-\alpha)}{\gamma \sinh \dfrac{\gamma b}{2}} \cdot \cosh \left[\gamma \left(x - a - \frac{b}{2} \right) \right] \cos \alpha z d\alpha \\ & \text{for } a \leq x \leq c \quad \text{and} \quad z \geq 0. \end{aligned} \tag{33}$$

Evaluating the foregoing integrals gives this final result:

$$
\begin{aligned}
H_y(x, z) &= \frac{a}{c}\left[1 + \frac{1}{K_+^2(k)} + \frac{i\nu_0}{4kK_-(k)} + \frac{i}{2K_+(k)} \sum_{m=1}^{\infty} \frac{\nu_{20}}{k + i\gamma_{2mb}}\right] e^{ikz} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \left[\sum_{m=1}^{\infty} \frac{\mu_{2m-1}}{\gamma_{(2n-1)b} + \gamma_{(2m-1)b}} \frac{1}{H_+(i\gamma_{(2n-1)b})} + \mu_{2n-1} \frac{d}{d\alpha} \frac{1}{H_+(\alpha)} \bigg|_{\alpha = i\gamma_{(2n-1)b}} \right] \right. \\
&\quad \left. \times \frac{2(-1)^n}{bc\gamma_{(2n-1)b}} \sin\left[\frac{(2n-1)\pi}{b}\left(x - a - \frac{b}{2}\right)\right] e^{-\gamma_{(2n-1)b}z} \right\} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \left[\frac{k + i\gamma_{2nb}}{2K_+(i\gamma_{2nb})} \sum_{m=0}^{\infty} \frac{\nu_{2m}}{\gamma_{2nb} + \gamma_{2mb}} + \frac{(i\gamma_{2nb} - k)\nu_{2n}}{2} \frac{d}{d\alpha} \frac{1}{K_-(\alpha)} \bigg|_{\alpha = i\gamma_{2nb}} \right. \right. \\
&\quad \left. \left. + \frac{2k}{K_+(k)K_+(i\gamma_{2nb})} \right] \frac{(-1)^n a}{ic\gamma_{2nb}} \cos\left[\frac{n\pi}{a}\left(x - a - \frac{b}{2}\right)\right] e^{-\gamma_{2nb}z} \right\} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \frac{1}{2n\pi i H_-'(i\gamma_{2nc})} \left[\sum_{m=1}^{\infty} \frac{\mu_{2m-1}}{\gamma_{(2m-1)b} - \gamma_{2nc}} \right] \frac{\sin\left[\frac{2n\pi}{c}\left(x - a - \frac{b}{2}\right)\right]}{\cos\left[\frac{n\pi}{c} b\right]} e^{-\gamma_{2nc}z} \right\} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \frac{-ab}{4n\pi K_-'(i\gamma_{2nc})} \left[\frac{2k}{K_+(k)} + \frac{k - i\gamma_{2nc}}{2} \sum_{m=0}^{\infty} \frac{\nu_{2m}}{\gamma_{2mb} - \gamma_{2nc}} \right] \frac{\cos\left[\frac{2n\pi}{c}\left(x - a - \frac{b}{2}\right)\right]}{\sin\left[\frac{n\pi}{c} b\right]} e^{-\gamma_{2nc}z} \right\}
\end{aligned}
\tag{34}
$$

for $a \le x \le c$ and $z \ge 0$.

Equations (30), (31), and (34) give the forms of the exact solution of the thick-wall problem, which is indeed very complicated. For practical purposes, it is sufficient to include only the propagating modes in the field expressions. Now let us assume that $c<\lambda/2$, i.e., the incident TEM mode is the only propagating mode inside the waveguides. Then the far field solution of the magnetic field can be greatly simplified, namely,

$$
H_y^{(t)}(x, z) = \frac{a}{c}\left[1 - \frac{ib}{4ka} \frac{\nu_0}{K_-(k)}\right] e^{ikz}, \quad \text{for } 0 \le x \le c,\ z \to +\infty \tag{35}
$$

$$
H_y(x, z) = \frac{ib}{4ka} \frac{\nu_0}{K_-(k)} e^{ikz} \quad \text{for } 0 \le x \le a,\ z \to -\infty \tag{36}
$$

where (27) has been used to eliminate the infinite summation over ν_{2m}. It is important to notice that (35) and (36) contain only one unknown constant, namely, ν_0. Thus one needs only to solve (27) for ν_0 alone in order to obtain the most interesting quantities, namely, the far field in the empty half-space, and the reflection coefficient inside the waveguides. Another interesting feature associated with (35) and (36) is that the transmission coefficient T and the reflection coefficient R are related exactly by $T=(a/c)(1-R)$. Hence, knowing R allows direct determination of the entire scattering matrix.

VI. On the Solutions of Linear Equations

First let us consider the set of simultaneous linear equations for unknown $\{\mu_{2n-1}\}$ in (24). Since the right-hand side of (24) is zero, the set $\{\mu_{2n-1}\}$ has a nonzero solution only when its coefficient determinant vanishes identically, i.e.,

$$
\begin{vmatrix}
\frac{bc}{4}\gamma_{1b}H_+^2(i\gamma_{1b}) + \frac{1}{2\gamma_{1b}} & \frac{1}{\gamma_{1b}+\gamma_{3b}} & \frac{1}{\gamma_{1b}+\gamma_{5b}} & \cdots \\
\frac{1}{\gamma_{3b}+\gamma_{1b}} & \frac{bc}{4}\gamma_{3b}H_+^2(i\gamma_{3b}) + \frac{1}{2\gamma_{3b}} & \frac{1}{\gamma_{3b}+\gamma_{5b}} & \cdots \\
\frac{1}{\gamma_{5b}+\gamma_{1}^{z}} & \frac{1}{\gamma_{5b}+\gamma_{3b}} & \frac{bc}{4}\gamma_{5b}H_+^2(i\gamma_{5b}) + \frac{1}{2\gamma_{5b}} & \cdots \\
\vdots & \vdots & \vdots &
\end{vmatrix} = 0. \tag{37}
$$

It should be noted that only the diagonal terms depend on c, while the rest of the terms depend on b. Since c and b can be varied independently, it is clear that (37) cannot hold for all values of c and b. Thus the only solution for (24) is

$$\mu_{2n-1} = 0, \qquad \text{for } n = 1, 2, 3, \cdots. \tag{38}$$

Consequently, all the terms with sine variation along x in (30), (31), and (34) are dropped. This is reasonable since the incident field is "symmetrical" and all of the "asymmetrical" field should not be excited.

It remains to consider the other set of simultaneous linear equations for ν_{2n} in (27). To the best of the author's knowledge, this set of equations cannot be solved exactly. Thus the infinite number of equations must be truncated at a proper finite number in order to obtain an approximated solution. As the first step, it is imperative to determine the convergence of the infinite set of equations or, equivalently, the asymptotic behavior of ν_{2n}. By definition,

$$\nu_{2n} = \frac{2(ik - \gamma_{2nb})}{b\gamma_{2nb}K_+(i\gamma_{2nb})} V_+'(i\gamma_{2nb}), \qquad \text{for } n = 1, 2, 3, \cdots. \tag{39}$$

It can be shown that

$$\gamma_{2nb} \sim O(2n)$$
$$K_+(i\gamma_{2nb}) \sim O(2n)^{1/2} \qquad \text{as } n \to \infty. \tag{40}$$

Furthermore, since $V_+'(\alpha)$ is proportional to E_z, the edge condition requires that

$$V_+'(i\gamma_{2nb}) \sim O(2n)^{-2/3} \qquad \text{as } n \to \infty. \tag{41}$$

It follows from (39) to (41) that

$$\nu_{2n} \sim O(2n)^{-7/6} \qquad \text{as } n \to \infty. \tag{42}$$

Therefore, the series in (27) converges absolutely. If one truncates the infinite set of equations at a finite number N, and sets $\nu_{2n}=0$ for $n>N$, the solution of these N equations will converge to the exact value as N increases indefinitely. Once ν_{2n} is obtained, (30), (31), and (34) would give the complete field solution. Therefore, in this sense, a method for solution exists.

So far, no restriction has been put on b, the wall thickness. For most practical cases, (b/λ) is a small number compared to unity. By making use of this fact, simplifications can be made in the solution of ν_0. Rewrite the first equation (27):

$$\left[\frac{icK_+^2(k)}{2ka} + \frac{i}{2k}\right]\nu_{2n0} + \sum_{n=1}^{\infty} \frac{\nu_{2n}}{-ik+\gamma_{2nb}} = \frac{-2}{K_+(k)}. \tag{43}$$

For small (b/λ) it can be shown that

$$K_+(k) \sim \left(\frac{b}{\lambda}\right)\Gamma\left(i\frac{b}{\lambda}\right)e^{ib \ln b}, \quad \text{as} \left(\frac{b}{\lambda}\right) \to 0 \tag{44}$$

or

$$|K_+(k)| \sim O(1).$$

Thus the coefficient of ν_0 in (43) is in the order of $O(1)$, and those of ν_{2n} for $n>0$ are $O(b/\lambda)$. It follows that

$$\nu_0 = \frac{i4ka}{K_+(k)[a + cK_+^2(k)]} + O\left(\frac{b}{\lambda}\right). \tag{45}$$

Substitution of (45) into (35) gives the far field

$$H_y^{(t)} = \frac{a}{c}\left[1 + \frac{b}{a + cK_+^2(k)}\right]e^{ikz} + O\left(\frac{b}{\lambda}\right)^2, \qquad \text{as } z \to \infty \tag{46}$$

and substitution of that into (36) gives the reflection coefficient

$$R = \frac{-b}{a + cK_+^2(k)} + O\left(\frac{b}{\lambda}\right)^2. \tag{47}$$

Therefore, the first-order solution of the far field and the reflection coefficient assume very simple forms. Furthermore, it is noted that $K_+(k)$ can be expressed in terms of tabulated functions, namely,

$$K_+(k) = \exp i\left[\frac{a}{\lambda}\ln a + \frac{b}{\lambda}\ln b - \frac{c}{\lambda}\ln c + S_1\left(\frac{a}{\lambda}, 0, 0\right) + S_1\left(\frac{b}{\lambda}, 0, 0\right) - S_1\left(\frac{c}{\lambda}, 0, 0\right)\right] \tag{48}$$

where

$$S_1(\mu, 0, 0) = \sum_{n=1}^{\infty} [\sin^{-1}(\mu/n) - (\mu/n)]$$

is tabulated by Marcuvitz [8]. Thus no computer work is required in calculating the numerical values of $H_y^{(t)}$ and R through (46) and (47). As a check on the accuracy of these two expressions, one may define an error term ϵ in the following manner:

$$\epsilon = \frac{P_i - (P_{\text{rad}} + P_{\text{ref}})}{P_i} \times 100 \text{ percent} \tag{49}$$

where P_i, P_{rad}, and P_{ref} are incident, radiated, and reflected powers, respectively. In this connection, it should be emphasized that even if $\epsilon=0$, it does not necessarily imply that the value of R is absolutely correct. Thus ϵ measures only the error committed in the power. A numerical example is given in Fig. 3 and Table I. For the case $(b/c)=(\frac{1}{4})$ which corresponds to a typical waveguide used in practice, the VSWR equals 1.35, with about 2 percent of the power reflected back. Therefore, wall thickness must be taken into consideration if a perfect match is desired.

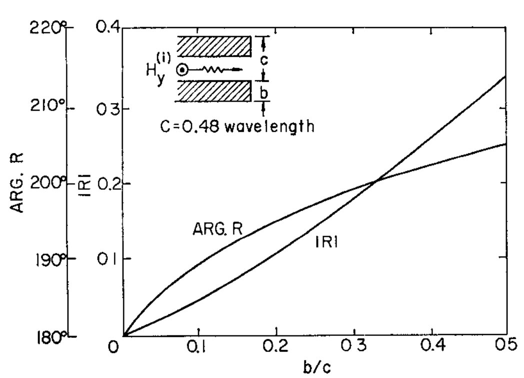

Fig. 3. Reflection coefficients of radiation from an infinite array of thick-wall waveguides.

TABLE I
REFLECTION COEFFICIENTS AND POWERS
($c=0.48\lambda$)

a/λ	b/λ	R	P_{rad}	P_{ref}	ϵ
0.43	0.05	$0.0557e^{i189.35°}$	0.994	0.004	+0.2%
0.40	0.08	$0.0925e^{i193.2°}$	0.989	0.008	+0.3%
0.36	0.12	$0.1480e^{i197.2°}$	0.986	0.022	−0.8%
0.30	0.18	$0.2408e^{i201.8°}$	0.937	0.058	+0.5%
0.24	0.24	$0.3470e^{i204.5°}$	0.878	0.120	+0.2%

VII. CONCLUSIONS AND GENERALIZATIONS

In this paper, the problem of broadside radiation from an infinite array of parallel-plate, thick-wall waveguides excited by incident TEM modes is investigated. For an arbitrary wall thickness the exact forms of field solutions are given by (30), (31), and (34) with constants ν_{2n} to be determined by an infinite set of simultaneous linear equations in (27). It is shown that ν_{2n} asymptotically decays as $(2n)^{-7/6}$, and therefore the infinite set of simultaneous linear equations can be truncated at a finite number for an approximate solution. For a small thickness of the wall, the explicit solutions of the far field and the reflection coefficient are given by (46) and (47), respectively, which are correct to the first order of (b/λ). By numerical examples it is shown that, for the commonly used waveguides, wall thickness does have an appreciable effect on impedance. Consequently, it must be accounted for in order to have a perect match.

Before concluding this paper, it is worth mentioning a few generalizations to the present problem.

1) By properly superimposing two incident TEM modes, the electric fields in both x and z directions may assume zero values at $y=0$, and d, and consequently two conducting planes may be placed at $y=0$ and d without causing any disturbance. Therefore, the solution in this paper obtained through a two-dimensional formulation may be applied to a three-dimensional problem and checked by experimentation.

2) Only the case with incident TEM modes is considered in this paper. Generalizations to an arbitrary incident wave can be achieved in a very similar manner. In particular, when the incident wave is of TE type, it practically gives the solution of an infinite array of rectangular waveguides [5].

3) An important and as yet unsolved problem is the radiation from a single waveguide covered with two ground planes at its opening. The question naturally arises whether it can be treated as a special case of the present problem when the wall thickness approaches infinity. Recall that the key point in solving (16) and (17) lies in the fact that $\tanh(\gamma b/2)/\gamma$ and $\coth(\gamma b/2)/\gamma$ have no branch singularities. In the limiting case when $b\to\infty$, this property is no longer preserved. However, Mittra [9] recently examined the transition from a function with poles to a function with branch points, and showed the analytic continuation from a closed-region problem to an open-region problem with success. Therefore, it is felt that the solution of the single waveguide problem may be obtained from the result in this paper although it is not obvious.

4) In this paper, only the broadside radiation is considered. For a scanning array where the main beam is pointed at an arbitrary direction in the empty half-space, there result two coupled Wiener-Hopf equations, which to the best of the author's knowledge cannot be solved by any known techniques. In this case, the numerical method used by Galindo and Wu [5] would be of value.

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REFERENCES

[1] J. F. Carlson and A. E. Heins, "The reflection of an electromagnetic wave by an infinite set of plates," *Quart. Appl. Math.*, vol. 4, pp. 313–329, January 1947.
[2] C. P. Wu and V. Galindo, "Properties of a phased array of rectangular waveguides with thin walls," *IEEE Trans. on Antennas and Propagation*, vol. AP-14, pp. 163–173, March 1966.
[3] D. J. Epstein, "Phase shift of microwaves in passage through parallel plate arrays," M.I.T., Lab. for Insulation Research, Cambridge, Mass., Tech. Rept. 42, August 1950.
[4] R. I. Primich, "A semi-infinite array of parallel metallic plates of finite thickness for microwave systems," *IEEE Trans. on Microwave Theory and Techniques*, vol. MTT-4, pp. 156–166, July 1956.
[5] V. Galindo and C. P. Wu, "Numerical solutions for an infinite phased array of rectangular waveguides with thick walls," *IEEE Trans. on Antennas and Propagation*, vol. AP-14, pp. 149–158, March 1966.
[6] D. S. Jones, "Diffraction by a thick semi-infinite plate," *Proc. Roy. Soc.* (London), vol. A-217, pp. 153–175, May 1953.
[7] B. Noble, *Methods Based on the Wiener-Hopf Technique*. New York: Macmillan, 1958, p. 6.
[8] N. Marcuvitz, *Waveguide Handbook*. New York: McGraw-Hill, 1951.
[9] R. Mittra, "Analytic continuation of closed region problem to open region problem," presented at the Internat'l Conf. on Microwaves, Circuit Theory, and Information Theory, Tokyo, Japan, 1964. See also "Solution of an open region problem as a limiting case of a closed region problem," *Appl. Sci. Research*, vol. 16, no. 3, 1966.
[10] S. W. Lee, "Radiation from an infinite array of parallel-plate waveguides with thick walls," Hughes Aircraft Co., Electromagnetics Lab., Fullerton, Calif., Tech. Rept., 1967.